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Pointwise Saturation of Positive Operators*

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1. This paper may be considered as a continuation of the papers of Amel'kovič [1], Mühlbach [17], and Lorentz and Schumaker [14] or, better, as a reinterpretation and extension of the first two papers with respect to the results obtained by Lorentz and Schumaker.

In the following two sections we formulate and prove a basic pointwise saturation theorem for sequences of positive operators, while the remaining sections are devoted to various applications.

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2. $C[a, b]$ denotes the space of all continuous real-valued functions $f(x)$ on the closed interval $[a, b]$ of the real axis.

Let v, w be functions in $C[a, b]$, strictly positive on (a, b), and let

$$\varphi(x) = \int_a^x v(t)\,dt, \qquad \psi(x) = \int_a^x w(t)\,dt$$

and

$$\Phi(x) = \int_a^x \psi(t)v(t)\,dt.$$

$\{1, \varphi, \Phi\}$ is a complete Chebychev system on $[a, b]$; see Karlin and Studden [10, Chap. XI].

For a function f in $C[a, b]$ we define the operation of differentiation $Df(x)$ at a point $x \in (a, b)$ by

$$Df(x) = D_\psi D_\varphi f(x) := \frac{1}{w(x)} \left[\frac{f'(x)}{v(x)}\right]',$$

whenever the right-hand side is meaningful.

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Obviously, the pair of functions $\{1, \varphi\}$ forms a complete system of solutions of

$$Df(x) = 0 \quad \text{on} \quad (a, b),$$

while

$$D\,\Phi(x) = 1 \quad \text{on} \quad (a, b).$$

Let $\{L_n : n = 1, 2,...\}$ be a sequence of positive linear transformations on $C[a, b]$ into itself, let $\{\lambda_n : n = 1, 2,...\}$ be a sequence of positive numbers tending to ∞ as $n \to \infty$, and let $\rho(x)$ be a function in $C[a, b]$, strictly positive on (a, b).

We say $\{L_n : n = 1, 2,...\}$ satisfies a *Voronovskaya condition* if and only if

$$\lim_{n\to\infty} \lambda_n\{L_n(f; x) - f(x)\} = \rho(x)\, Df(x), \qquad x \in (a, b), \tag{1}$$

whenever $Df(x)$ exists.

Trivially, for each $x \in (a, b)$,

$$\lambda_n\{L_n(1; x) - 1\} \to 0, \qquad \lambda_n\{L_n(\varphi; x) - \varphi(x)\} \to 0,$$

while

$$\lambda_n\{L_n(\Phi; x) - \Phi(x)\} \to \rho(x),$$

as $n \to \infty$. Moreover, by Korovkin's theorem [11, p. 46],

$$\lim_{n\to\infty} L_n(f; x) = f(x) \qquad (f \in C[a, b])$$

pointwise for each $x \in (a, b)$.

THEOREM 1. *Let $\{L_n : n = 1, 2,...\}$ be a sequence of positive operators on $C[a, b]$ into itself which satisfies condition* (1). *Let G be a function in $C[a, b]$, and let g be a finitely-valued Lebesgue-integrable function on (a, b) such that for each $x \in (a, b)$*

$$\liminf_{n\to\infty} \lambda_n\{L_n(G; x) - G(x)\} \leqslant \rho(x)\, g(x) \leqslant \limsup_{n\to\infty} \lambda_n\{L_n(G; x) - G(x)\}. \tag{2}$$

Then there are two constants A and B such that

$$G(x) = A + B\varphi(x) + \int_a^x v(t)\, dt \int_a^t g(u)\, w(u)\, du \tag{3}$$

on $[a, b]$.

This is a *pointwise saturation theorem* for the sequence $\{L_n : n = 1, 2,...\}$.

It is a converse of Voronovskaya's condition (1). Indeed, if the function G in $C[a, b]$ is given by (3), then for almost all x in (a, b)

$$\lim_{n\to\infty} \lambda_n\{L_n(G; x) - G(x)\} = \rho(x)\, g(x). \tag{4}$$

Moreover, the theorem is best possible in the following sense: If condition (2) is violated at even one point in (a, b) then the satement of Theorem 1 is wrong. Indeed, setting

$$f_0(x) = |\,\varphi(x) - \varphi(x_0)|, \qquad x_0 \in (a, b),$$

and we have for all $x \in (a, b)$, $x \neq x_0$,

$$\lim_{n\to\infty} \lambda_n\{L_n(f_0\,; x) - f_0(x)\} = 0,$$

while at $x = x_0$

$$\lim_{n\to\infty} \lambda_n\{L_n(f_0\,; x_0) - f_0(x_0)\} = +\infty.$$

Remark. The theorem may be looked at as a generalization of the following fundamental lemma of de la Vallée-Poussin in the theory of trigonometric series:

Let G belong to $C[a, b]$, and let g be a finitely-valued, L-integrable function on (a, b) such that for all x in (a, b)

$$\underline{D}_2G(x) \leqslant g(x) \leqslant \overline{D}_2G(x).$$

Then

$$G(x) = A + Bx + \int_a^x dt \int_a^t g(u)\, du, \qquad a \leqslant x \leqslant b,$$

where A and B are two constants.

The expressions $\underline{D}_2 f(x)$ and $\overline{D}_2 f(x)$ are the lower and upper second symmetric derivatives of f at x, respectively.

Indeed, if we define

$$L_t(f; x) = \frac{f(x+t) + f(x-t)}{2}, \qquad t > 0,$$

then we have

$$\lim_{t\to 0+} t^{-2}\{L_t(f; x) - f(x)\} = \tfrac{1}{2} f''(x),$$

whenever $f''(x)$ exists.

For this set of operators, Theorem 1 reduces to de la Vallée-Poussin's lemma. Moreover, the proof of Theorem 1 is basically a transformation of the arguments in de la Vallée-Poussin's proof (cf. Hardy and Rogosinski [9, p. 90]). The relationship of de la Vallée-Poussin's lemma with the phenomenon of saturation was pointed out by Butzer in several papers; see, in particular, [5].

As a consequence of Theorem 1 we obtain the so-called *saturation theorem* for $\{L_n : n = 1, 2,...\}$.

THEOREM 2. *Let* $\{L_n : n = 1, 2,...\}$ *be a sequence of positive operators on* $C[a, b]$ *into itself which satisfies condition* (1), *and let* G *belong to* $C[a, b]$. *If for all* x *in* (a, b)

$$\lambda_n \mid L_n(G; x) - G(x)| \leqslant M\rho(x) + o_x(1) \tag{5}$$

for some positive constant M, *then* $D_\varphi G(x)$ *exists, belongs to* $C[a, b]$, *and*

$$| D_\varphi G(y) - D_\varphi G(x)| \leqslant M \mid \psi(y) - \psi(x)| \tag{6}$$

for all $x, y \in [a, b]$, *and vice versa.*

Remark. In its given form Theorem 2 is due to Lorentz and Schumaker [14], who even formulated and proved a more general version. A more special version of Theorem 2 can be found in Amel'kovič's and Mühlbach's papers. At this point it has to be mentioned that it was Amel'kovič who gave the first analytic proof of Lorentz's saturation theorem for the Bernstein polynomials (cf. Lorentz [13, p. 102ff.]). Lorentz's original proof is essentially a functional-analytic one; his method has also been applied for proving local saturation theorems for various sequences of positive operators, see in particular Suzuki [19] and Suzuki and Watanabe [20].

In one of the following applications the weight functions v and w in $C[a, b]$ are strictly negative on (a, b). Setting

$$\varphi(x) = -\int_x^b v(t)\,dt \qquad \text{and} \quad \psi(x) = -\int_x^b w(t)\,dt,$$

the two theorems remain valid with (3) replaced by

$$G(x) = A + B\varphi(x) + \int_x^b v(t)\,dt \int_t^b g(u)\,w(u)\,du. \tag{3'}$$

3. For the proof of Theorem 1 we need three lemmas. But before we can formulate these lemmas, we have to give the following definition:

A function $f \in C[a, b]$ is said to be *convex (concave) with respect to* $\{1, \varphi\}$ on $[a, b]$ if and only if

$$\{\varphi(x_1) - \varphi(x_0)\} f(x) \leqslant (\geqslant) \{\varphi(x) - \varphi(x_0)\} f(x_1) + \{\varphi(x_1) - \varphi(x)\} f(x_0)$$

for all $x, x_0, x_1 \in [a, b]$ where $x_0 \leqslant x \leqslant x_1$. A function which is both convex and concave with respect to (w.r.t.) $\{1, \varphi\}$ is said to be *linear* w.r.t. $\{1, \varphi\}$ [10].

LEMMA 1. *The function $f \in C[a, b]$ is convex w.r.t. $\{1, \varphi\}$ on $[a, b]$ if and only if*

$$0 \leqslant \limsup_{n\to\infty} \lambda_n\{L_n(f: x) - f(x)\} \qquad on \quad (a, b). \tag{6}$$

Remark. A proof of this lemma can be found in the papers of Mühlbach and Lorentz and Schumaker. For the Bernstein polynomials it goes back to Bajšanski and Bojanić [2]. Actually, the method of proof of Lemma 1 has its origin in the proof of de la Vallée-Poussin's lemma (cf. Hardy and Rogosinski [9, p. 87]). In this connection we also have to mention the paper [8] of DeVore who studies among others the saturation problem for sequences of positive linear operators on $C[-1, 1]$ into itself of so-called optimal convergence by using these methods.

COROLLARY. *If for a function $f \in C[a, b]$*

$$\liminf_{n\to\infty} \lambda_n\{L_n(f; x) - f(x)\} = 0$$

for each x in (a, b), then f is linear w.r.t. $\{1, \varphi\}$ *on* $[a, b]$.

This is the so-called pointwise "o"-theorem for the sequence $\{L_n : n = 1, 2,...\}$. It means that for an $f \in C[a, b]$ the order of approximation of $f(x)$ by $L_n(f; x)$ on (a, b) is at most of order $O_x(1/\lambda_n)$ $(n \to \infty)$ unless $f(x)$ is linear w.r.t. $\{1, \varphi\}$.

LEMMA 2. *Let $f \in C[a, b]$, and let $F(x) = \int_a^x f(t)\, v(t)\, dt$. For each $x \in (a, b)$*

$$\rho(x)\underline{D}_\psi f(x) \leqslant \overline{\underline{\lim_{n\to\infty}}}\, \lambda_n\{L_n(F; x) - F(x)\} \leqslant \rho(x)\bar{D}_\psi f(x). \tag{7}$$

Proof. We have

$$\lambda_n\{L_n(F; x) - F(x)\} = \lambda_n L_n(F(\cdot) - F(x) - \{\varphi(\cdot) - \varphi(x)\} f(x); x) + o_x(1).$$

Thus it is enough to show that

$$\rho(x)\underline{D}_\psi f(x) \leqslant \overline{\lim_{n\to\infty}}\, \lambda_n L_n(F(\cdot) - F(x) - \{\varphi(\cdot) - \varphi(x)\} f(x); x)$$
$$\leqslant \rho(x)\overline{D}_\psi f(x). \tag{7'}$$

We shall only prove the right-hand side of this inequality. Introducing

$$\Phi(y) - \Phi(x) - \{\varphi(y) - \varphi(x)\}\, \psi(x) = \int_x^y v(t)\, dt \int_x^t w(u)\, du \begin{cases} > 0, & y \neq x, \\ = 0, & y = x, \end{cases}$$

we obtain

$$\frac{F(y) - F(x) - \{\varphi(y) - \varphi(x)\} f(x)}{\Phi(y) - \Phi(x) - \{\varphi(y) - \varphi(x)\}\, \psi(x)} = \frac{\int_x^y \frac{f(t) - f(x)}{\psi(t) - \psi(x)}\, v(t)\, dt \int_x^t w(u)\, du}{\int_x^y v(t)\, dt \int_x^t w(u)\, du}$$

from which it easily follows that

$$\underline{D}_\psi f(x) \leqslant \overline{\lim_{y\to x}}\, \frac{F(y) - F(x) - \{\varphi(y) - \varphi(x)\} f(x)}{\Phi(y) - \Phi(x) - \{\varphi(y) - \varphi(x)\}\, \psi(x)} \leqslant \overline{D}_\psi f(x).$$

For $\overline{D}_\psi f(x) = +\infty$, there is nothing to prove.

Let $\overline{D}_\psi f(x) = d$, finite. Given an $\epsilon > 0$, there is a $\delta = \delta(x, \epsilon) > 0$ such that

$$F(y) - F(x) - \{\varphi(y) - \varphi(x)\} f(x)$$
$$\leqslant (d + \epsilon)(\Phi(y) - \Phi(x) - \{\varphi(y) - \varphi(x)\}\, \psi(x)),$$

whenever $| y - x | < \delta$, and, consequently,

$$\lambda_n L_n(F(\cdot) - F(x) - \{\varphi(\cdot) - \varphi(x)\} f(x); x)$$
$$\leqslant (d + \epsilon)\, \lambda_n L_n(\Phi(\cdot) - \Phi(x) - \{\varphi(\cdot) - \varphi(x)\}\, \psi(x); x) + o_x(1).$$

Hence

$$\limsup_{n\to\infty}\, \lambda_n L_n(F(\cdot) - F(x) - \{\varphi(\cdot) - \varphi(x)\} f(x); x) \leqslant \rho(x)(d + \epsilon)$$

for every $\epsilon > 0$, which gives the desired result.

For $\overline{D}_\psi f(x) = -\infty$, one obtains similarly

$$\limsup_{n\to\infty} \lambda_n L_n(F(\cdot) - F(x) - \{\varphi(\cdot) - \varphi(x)\} f(x); x\} \leqslant \rho(x) K$$

for any real number K.

The final lemma is a well-known result from the theory of the Lebesgue integral due to de la Vallée-Poussin (cf. Hardy and Rogosinski [9, p. 90]).

LEMMA 3. *Let $g \in L(a, b)$. Then there are sequences $\{p_m : m = 1, 2,...\}$ and $\{P_m : m = 1, 2,...\}$ of functions in $C[a, b]$ such that*

(i) $p_k(a) = P_m(a) = 0$;
(ii) $p_m(x), P_m(x) \to \int_a^x g(u)\, w(u)\, du$ *uniformly on* $[a, b]$;
(iii) $\bar{D}_\psi p_m(x) \leqslant g(x) \leqslant \underline{D}_\psi P_m(x)$ *in* (a, b), *whenever* $g(x)$ *is finite.*

Proof of Theorem 1. Let

$$f(x) = \int_a^x g(t)\, w(t)\, dt \qquad \text{and} \quad F(x) = \int_a^x f(t)\, v(t)\, dt,$$

and let $\{p_m : m = 1, 2,...\}$ and $\{P_m : m = 1, 2,...\}$ be the sequences of functions associated to g via Lemma 3. We set

$$q_m(x) = \int_a^x p_m(t)\, v(t)\, dt \qquad \text{and} \quad Q_m(x) = \int_a^x P_m(t)\, v(t)\, dt.$$

Finally we define

$$K(x) = F(x) - G(x); K_m(x) = Q_m(x) - G(x); k_m(x) = q_m(x) - G(x).$$

The theorem is proved if we can show that $K(x)$ is linear w.r.t. $\{1, \varphi\}$ on $[a, b]$.

For each fixed x in (a, b), we obtain by Lemmas 2 and 3

$$\begin{aligned} &\limsup_{n\to\infty} \lambda_n\{L_n(K_m ; x) - K_m(x)\} \\ &\qquad \geqslant \liminf_{n\to\infty} \lambda_n\{L_n(Q_m ; x) - Q_m(x)\} - \liminf_{n\to\infty} \lambda_n\{L_n(G; x) - G(x)\} \\ &\qquad \geqslant \rho(x)\{\underline{D}_\psi P_m(x) - g(x)\} \geqslant 0. \end{aligned}$$

By Lemma 1, $K_m(x)$ is convex w.r.t. $\{1, \varphi\}$ on $[a, b]$ for each m. Likewise one proves that k_m is concave w.r.t. $\{1, \varphi\}$ on $[a, b]$ for each m. Taking into account that both $K_m(x)$ and $k_m(x)$ converge to $K(x)$ uniformly on $[a, b]$ as $m \to \infty$, and we have that $K(x)$ is linear w.r.t. $\{1, \varphi\}$ on $[a, b]$. This proves the theorem.

Proof of Theorem 2. We define the function g in Theorem 1 by

$$\rho(x)\, g(x) = \liminf_{n\to\infty} \lambda_n\{L_n(G; x) - G(x)\} \qquad \text{on} \quad (a, b).$$

By Theorem 1, $G(x)$ is then given via (3) with $| g(x) | \leqslant M$ on (a, b), which proves the "if" part.

To prove the "only if" part, remember that

$$\lambda_n\{L_n(G; x) - G(x)\}$$
$$= \lambda_n L_n(G(\cdot) - G(x) - \{\varphi(\cdot) - \varphi(x)\} D_\varphi G(x); x) + o_x(1)$$

and that

$$| G(y) - G(x) - \{\varphi(y) - \varphi(x)\} D_\varphi G(x) |$$
$$= \left| \int_x^y \{D_\varphi G(t) - D_\varphi G(x)\} v(t)\, dt \right|$$
$$\leqslant M\{\Phi(y) - \Phi(x) - \{\varphi(y) - \varphi(x)\} \psi(x)\}$$

giving

$$\lambda_n \mid L_n(G; x) - G(x) \mid \leqslant \rho(x)M + o_x(1) \qquad (n \to \infty).$$

4. As the first application we study the Bernstein polynomials

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1 - x)^{n-k} \qquad (n = 1, 2,...; f \in C[0, 1]).$$

For these polynomials Voronovskaya proved [12, p. 22] that

$$\lim_{n\to\infty} n\{B_n(f; x) - f(x)\} = \frac{x(1 - x)}{2} f''(x) \qquad (0 < x < 1)$$

whenever $f''(x)$ exists.

In this case, $v(x) = w(x) = 1$, $\rho(x) = x(1 - x)/2$ on $[0, 1]$, and $\{\lambda_n = n : n = 1, 2,...\}$. Theorem 1 then reads:

Let f belong to $C[0, 1]$, and let g be a finitely-valued, L-integrable function on some subinterval (a, b) in $[0, 1]$ such that for all $x \in (a, b)$

$$\liminf_{n\to\infty} n\{B_n(f; x) - f(x)\} \leqslant \frac{x(1 - x)}{2} g(x) \leqslant \limsup_{n\to\infty} n\{B_n(f; x) - f(x)\}.$$

Then

$$f(x) = A + Bx + \int_a^x dt \int_a^t g(u)\, du \qquad \text{on} \quad [a, b],$$

where A and B are two suitable constants.

For the associated saturation theorem, see Lorentz [13, p. 102].

For the Bernstein power series

$$P_n(f; x) = (1 - x)^n \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k-1}{k} x^k \quad (n=1,2,...; f \in CB[0, 1))$$

($CB[0, 1)$ is the space of all bounded continuous functions on $[0, 1)$), introduced by Meyer-König and Zeller [16], the following Voronovskaya condition is known: Setting $\varphi(x) = x/(1 - x)$,

$$\lim_{n\to\infty} n\{P_n(f; x) - f(x)\} = \frac{\varphi(x)[1 + \varphi(x)]}{2} D_\varphi^2 f(x), \qquad 0 < x < 1,$$

whenever $D_\varphi^2 f(x)$ exists.

Here we have $v(x) = w(x) = 1/(1 - x)^2$, $\rho(x) = \varphi(x)[1 + \varphi(x)]/2$, and $\{\lambda_n = n : n = 1, 2,...\}$. We only want to formulate the local saturation theorem:

(a) *If for an* $f \in CB[0, 1]$

$$\liminf_{n\to\infty} n\{P_n(f; x) - f(x)\} = 0$$

on some subinterval (a, b) *in* $[0, 1)$ *with* $b < 1$, *then* $f(x)$ *is linear* w.r.t. $\{1, \varphi\}$.

(b) *Let* $f \in CB[0, 1)$. *If there exists a positive constant M such that for all* $x \in (a, b) \subset [0, 1)$, $b < 1$,

$$n \mid P_n(f; x) - f(x) \mid \leqslant \frac{x}{(1 - x)^2} \frac{M}{2} + o_x(1) \qquad (n = 1, 2,...)$$

holds true, then $f' \in C[a, b]$ *and for any pair* $x, y \in [a, b]$

$$\mid (1 - y)^2 f'(y) - (1 - x)^2 f'(x) \mid \leqslant \frac{M}{(1 - y)(1 - x)} \mid y - x \mid,$$

and vice versa.

The Bernstein power series have been generalized by Cheney and Sharma [7]. For the saturation theorem for these generalized Bernstein power series, see Lorentz and Schumaker [14].

As a third application we investigate the following sequence $\{M_n : n = 1, 2,...\}$ of operators on $CB[0, \infty)$ into itself constructed by Baskakov [3]: Let $\{\varphi_n : n = 1, 2,...\}$ be a sequence of completely monotonic functions on $[0, \infty)$ such that (i) $\varphi_n(0) = 1$ and (ii) $-\varphi_n^{(k)}(x) = n\varphi_{n+c}^{(k-1)}(x)$, $k = 1, 2,...$ and c a positive integer. The sequence $\{M_n : n = 1, 2,...\}$ is then defined by

$$M_n(f; x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(-1)^k \varphi_n^{(k)}(x)}{k!} x^k \qquad (n = 1, 2,...; f \in CB[0, \infty)),$$

and it satisfies the following Voronovskaya condition:

$$\liminf_{n\to\infty} n\{M_n(f; x) - f(x)\} = \frac{x(1 + cx)}{2} f''(x),$$

whenever $f''(x)$ exists.

Here $v(x) = w(x) = 1$, $\rho(x) = x(1 + cx)/2$ and $\{\lambda_n = n : n = 1, 2,...\}$.

Examples are $\varphi_n(x) = e^{-nx}$ with $c = 0$ and $\varphi_n(x) = (1 + x)^{-n}$ with $c = 1$, which lead to the Szasz operators

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!}$$

and via the transformation $x \to y/(1 - y)$ to the Bernstein power series of Meyer–König and Zeller [16], respectively.

The local saturation theorem for the Baskakov operators $\{M_n : n = 1, 2,...\}$ is due to Suzuki [19].

5. As the final two applications we discuss the Gauss–Weierstrass operators $\{W_t : t > 0\}$ on $CB(-\infty, \infty)$ and the Gamma operators $\{G_n : n = 1, 2,...\}$ on $CB(0, \infty)$.

For an $f \in CB(-\infty, \infty)$, $W_t(w; x)$ is defined by

$$W_t(f; x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(x - u) \exp(-u^2/4t)\, du.$$

It is known [4], that

$$\lim_{t\to 0+} \frac{W_t(f; x) - f(x)}{t} = f''(x),$$

whenever $f''(x)$ exists.

We formulate the local saturation theorem:

(a) *If for an f in $CB(-\infty, \infty)$*

$$\liminf_{t\to 0+} \frac{W_t(f; x) - f(x)}{t} = 0$$

on some finite interval (a, b), then $f(x)$ is linear on $[a, b]$.

(b) *For an $f \in CB(-\infty, \infty)$, the following are equivalent:*

(i) $| W_t(f; x) - f(x) | \leqslant Mt + o_x(1)$ *on (a, b),*

(ii) *$f'(x)$ exists on $[a, b]$ and $f' \in \mathrm{Lip}_M(1; C[a, b])$.*

For a discussion of the approximation behavior of the Gauss–Weierstrass operator see the monograph of Butzer and Nessel [6].

The Gamma operators on $CB(0, \infty)$ are defined by

$$G_n(f; x) = \int_0^\infty f(nu)\, g_n\left(\frac{x}{u}\right) \frac{du}{u},$$

where $g_n(u) = u^n e^{-u}/(n-1)!$. They were introduced by Feller, their approximation behavior was intensively studied by Müller [18] and Lupas and Müller [15].

Setting $\varphi(x) = 1/x$, and it is easy to verify that

$$\lim_{n\to\infty} n\{G_n(f; x) - f(x)\} = \frac{\varphi^2(x)}{2} D_\varphi^2 f(x),$$

whenever $D_\varphi^2 f(x)$ exists.

Here, $v(x) = w(x) = -1/x^2$, $\rho(x) = 1/2x^2$ and $\{\lambda_n = n : n = 1, 2, \ldots\}$. We have

(a) *Let $f \in CB(0, \infty)$, and let g be a finitely-valued, L-integrable function on some subinterval (a, b) of $(0, \infty)$ with $a > 0$. If for all $x \in (a, b)$*

$$\liminf_{n\to\infty} n\{G_n(f; x) - f(x)\} \leqslant \frac{1}{2x^2} g(x) \leqslant \limsup_{n\to\infty} n\{G_n(f; x) - f(x)\},$$

then

$$f(x) = A + \frac{B}{x} + \int_x^b \frac{dt}{t^2} \int_t^b g(u) \frac{du}{u^2} \qquad on \quad [a, b]$$

for some constants A and B.

(b) *Let $f \in CB(-\infty, \infty)$. If there exists a positive constant M such that*

$$| G_n(f; x) - f(x) | \leqslant \frac{M}{2x^2} + o_x(1)$$

on some subinterval $(a, b) \subset (0, \infty)$ with $a > 0$, then $f'(x)$ exists on $[a, b]$ and

$$| y^2 f'(y) - x^2 f'(x) | \leqslant \frac{M}{xy} | y - x |, \qquad x, y \in [a, b],$$

and vice versa.

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